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Field Theory of the Two-Dimensional Ising Model: Equivalence to the Free Particle One-Dimensional Dirac Equation

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The Schultz–Mattis–Lieb fermion formulation of the two-dimensional Ising model is simplified by means of long-wavelength approximations which become exact in the critical region. The resulting continuum theory has a Hamiltonian density which is shown to be identical, to within a perfect derivative, to that of free, spinless particles satisfying the one-dimensional Dirac equation. Filling the negative-energy single-particle states of momentum q and mass κ gives an integral over the single-particle energies $-(q^2 + \kappa^2)^{1/2}$. Because κ varies linearly with the temperature, differentiating twice gives Onsager's logarithmic singularity in the specific heat.

KEY WORDS: Field theory; Ising model; Dirac equation; continuum limit; logarithmic specific heat; critical behavior; phase transition.

1. INTRODUCTION

The Ising model[2] is of fundamental importance for understanding the phenomenon of phase transitions. An exact solution of the model in three-

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[2] For a survey of the history of the Ising model, see Ref. 1.

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dimensional space has never been achieved, so one depends very much upon Onsager's[2] solution of the two-dimensional version. (The even simpler one-dimensional version, as we shall see shortly, does not exhibit a sharp phase transition.) Although Onsager's treatment was simplified by Kaufman,[3] it remains a formidable tour de force of spinor algebra. Although impressive, this is nevertheless regrettable in view of the central position this solution holds in the field of phase transitions. It would be useful to have available an exposition of it in a form which could be easily followed by an intermediate student in statistical mechanics and many-body theory. To attempt to fill this gap is the goal of this paper.

There is, in fact, an alternative approach to the two-dimensional Ising model. This combinatorial method[4–6] possesses considerable elegance and has recently been reviewed by Glasser.[7] However, it seems to have no direct connection with the present mainstream of the field-theoretical formulation of the problem of phase transition.[8–11] Therefore, we prefer to adhere essentially to the techniques used by Onsager and Kaufman. This is the method of the transfer matrix, which is basically a way of working through the system under study, progressing in a certain direction. Thus, the use of the transfer matrix is similar to the use of partial differential equations in field theory, where the coordinate describing the progression is the time variable t. Knowing the initial values (and derivatives) of the field variable at time t, we can apply the appropriate Green's function (i.e., a suitable solution of the partial differential equation) to find their values at a slightly greater time $t + \Delta t$. The method of the transfer matrix works in an analogous way.

Fortunately, the transfer matrix approach has been greatly simplified by Schultz *et al.*[12] They have noted that the dichotomy in the Ising spin variable ("up" or "down") can be transformed to a "yes-or-no" question referring to the presence or absence of certain fictitious particles. In order to avoid the undesired possibility of multiple occupation of a site by these particles, they are assumed to be fermions and to obey the Pauli exclusion principle. In the present paper, we try to present the Schultz–Mattis–Lieb transform in its simplest possible form. We strip the notation bare and concentrate on those features of it that are essential for obtaining the Onsager logarithmic singularity in the specific heat. Other aspects of the formalism, which are necessary for calculating the spontaneous magnetization[13] or the correlation functions, are omitted here.

The enormous simplification achieved by Schultz *et al.* is that the fictitious particles that they introduce are governed by a very simple interaction: This is one corresponding to the presence of a field which spontaneously emits and absorbs pairs of particles possessing zero net total momentum. This is the same kind of effect encountered in the BCS[14] theory

of superconductivity and can easily be handled by the methods developed for that subject. But rather than present the usual calculation of this pairing effect, we further simplify the problem in Section 4 by an additional transformation. This reduces the problem to that of completely free fermions satisfying the one-dimensional Dirac equation. In Section 5, we fill all of the negative-energy states and sum up to get the total ground-state energy of the system. This leads immediately to the Onsager formula for the logarithmic singularity in the specific heat.

Let us begin with the one-dimensional version of the Ising model. We measure the temperature T in natural units, so that Boltzmann's constant is unity, and we denote the exchange coefficient by TJ'. Thus, the energy of interaction of the spins at the neighboring sites j and $j+1$ is $-TJ'\sigma_j\sigma_{j+1}$. The corresponding Boltzmann factor is $\exp(J'\sigma_{j+1}\sigma_j)$ and the partition function is

$$Z_N = \sum_{\{\sigma_j\}} \prod_j \exp(J'\sigma_{j+1}\sigma_j) \tag{1}$$

Every one of the N spin variables σ_j is summed over the two possible values ± 1. Now let us denote the partial product summed over all spins $j' < j$ by

$$\psi_j(\sigma_j) = \sum_{\{\sigma_{j'}\}} \prod_{j'<j} \exp(J'\sigma_{j'+1}\sigma_{j'}) \tag{2}$$

Replacing j in Eq. (2) by $j+1$ gives by inspection

$$\psi_{j+1}(\sigma_{j+1}) = \sum_{\sigma_j} \mathscr{T}_{\sigma_{j+1}\sigma_j}\psi_j(\sigma_j) \tag{3}$$

where the 2×2 transfer matrix is

$$\mathscr{T} = \begin{pmatrix} \exp J' & \exp -J' \\ \exp -J' & \exp J' \end{pmatrix} \tag{4}$$

with eigenvalues

$$\lambda_\pm = \exp J' \pm \exp -J' = 2\begin{cases}\cosh J' \\ \sinh J'\end{cases} \tag{5}$$

and associated eigenfunctions

$$\psi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{6}$$

The larger eigenvalue λ_+, associated with the symmetric eigenfunction ψ_+ dominates Eq. (1) for large N. Defining the effective partition function per spin by $Z_1 = Z_N^{1/N}$ and taking the thermodynamic limit $N \to \infty$ gives

$$Z_1 = \lim_{N\to\infty} Z_N^{1/N} = \lambda_+ \tag{7}$$

The free energy and entropy per site are consequently

$$F = -T \ln \lambda_+ \tag{8}$$

and

$$S = -\frac{\partial F}{\partial T} = \ln \lambda_+ + \frac{d \ln \lambda_+}{d \ln T} \tag{9}$$

Differentiating again gives the specific heat

$$C = T\frac{\partial S}{\partial T} = \frac{\partial \ln \lambda_+}{\partial \ln T} + \frac{d^2 \ln \lambda_+}{(d \ln T)^2} = -\frac{d \ln \lambda_+}{d \ln J'} + \frac{d^2 \ln \lambda_+}{(d \ln J')^2} = J'^2 \frac{d^2 \ln \lambda_+}{dJ'^2} \tag{10}$$

where the exchange constant TJ' is assumed to be independent of temperature, so that $d \ln T = -d \ln J'$. Substitution from Eq. (5) yields

$$C = J'^2 \operatorname{sech}^2 J' \tag{11}$$

which exhibits a broad maximum in the vicinity of $J'_{\max}$ determined by

$$J'_{\max} \tanh J'_{\max} = 1 \qquad \text{or} \qquad J'_{\max} = 1.20 \tag{12}$$

2. FERMION FORMALISM

The horizontal and vertical coordinates in the two-dimensional Ising lattice will be denoted by the indices i and j, respectively. Continuing to write the exchange constant in the vertical direction as TJ', and writing the exchange constant for a horizontal bond as TJ, we have the total energy of the lattice composed of two double summations over the entire Ising lattice,

$$E = -TJ \sum_{ij} \sigma_{ij}\sigma_{i+1,j} - TJ' \sum_{ij} \sigma_{ij+1}\sigma_{ij} \tag{13}$$

The transfer matrix for the vertical step $j \to j + 1$ is simply the direct product over the index i of all of the one-dimensional Ising model transfer matrices. Each of these individual transfer matrices has the eigenvalues $2 \cosh J'$ and $2 \sinh J'$, associated with the "ground and excited" states of a vertical bond. Thus, if n is the total number of "excitations" and L the horizontal width of the lattice, the $j \to j + 1$ transfer matrix is

$$(2 \cosh J')^L M' = (2 \cosh J')^L \tanh^n J' \tag{14}$$

where

$$M' = \tanh^n J' = \exp(-2n\bar{J}') \tag{15}$$

It has been convenient to introduce the new parameter $\bar{J}'$ defined by

$$\exp -2\bar{J}' = \tanh J' \tag{16}$$

which can be put into the more symmetric form

$$\sinh 2\bar{J}' \cdot \sinh 2J' = 1 \tag{17}$$

Because the ground-state eigenvalue has been factored off, M' represents the additional critical behavior exhibited by the two-dimensional Ising model over and above the noncritical one-dimensional model. Thus, for example, to the critical specific heat calculated below has to be added that of the one-dimensional model, Eq. (11), as background.

We now have to study the portion of the transfer matrix which describes the crossing of the horizontal lines as we work our way upward in the Ising lattice. This involves the first double sum in Eq. (13). Since we will be working at a definite horizontal line, say the $(j+1)$th line, we will suppress the $j+1$ subscript and will indicate the operators simply by the index identifying the vertical line on which they act. The individual spin operators can excite or deexcite one of the vertical lines. Following Schulz *et al.*,[12] we can describe this by creation and annihilation operators in a second quantized Fermi–Dirac formalism. The equivalence is given by

$$\sigma_{i,j+1}\sigma_{i+1,j+1} = c_i^{\dagger}c_{i+1}^{\dagger} + c_{i+1}c_i + c_i^{\dagger}c_{i+1} + c_{i+1}^{\dagger}c_i \tag{18}$$

The four terms in Eq. (18) describe the four possibilities of creation or annihilation by either one of the neighboring pair of spins. In expressing the spin operators in this way, it is important to realize that the correspondence will only be correct if there is agreement between the transition matrix elements between all pairs of states as calculated in the two different formalisms. In the Fermi–Dirac notation, we can imagine that the basis states are products of creation operators ordered in a definite way, say with smaller subscripts to the left. Thus, it is quite important to establish the correct order of the anticommuting operators in Eq. (18). The first term, for example, is written so that the pair of excitations produced at sites i and $i+1$ will be in the standard order. Similarly, the second term is written so that the matrix element of it between the two states in which the i and $i+1$ lines are excited and deexcited, respectively, is unity. The order chosen is such that the annihilation operator for site i works first, so as to remove the i-excitation. The annihilation operator associated with site $i+1$ then follows. Similarly, the third term describes the replacement of an excitation at site $i+1$ by by an excitation at site i, or in other words, an exchange of the excitation from one line to the other. Similarly, the last term describes the reverse exchange. Special attention has to be given to the cyclic condition which applies at

the end of the chain. We imagine that the Ising lattice in the direction of increasing i closes itself after the maximum value of $i = L - 1$ is reached; i.e., $i = L$ is identical to $i = 0$. Thus, we have the identity $\sigma_{L,j} \equiv \sigma_{0,j}$, which requires

$$\sigma_{L-1,j+1}\sigma_{L,j+1} \equiv \sigma_{L-1,j+1}\sigma_{0,j+1} = c_0^{\dagger}c_{L-1}^{\dagger} + c_{L-1}c_0 - c_{L-1}^{\dagger}c_0 - c_0^{\dagger}c_{L-1} \tag{19}$$

The terms describing the exchange of excitation now have a negative sign associated with the reordering which is required for the creation operators. Here we are assuming that we are dealing with a state having an even number of excitations. (The interaction described by the horizontal bonds leaves the oddness or evenness of the states unchanged. Consequently, there is a kind of parity operator which is a good quantum number in the present problem.) After the annihilation operator has acted, an odd number of operator exchanges is required in order to bring the remaining creation operator into its correct standard position. It is convenient to change the order in the first two terms of Eq. (19), which gives

$$\sigma_{L-1,j+1}\sigma_{L,j+1} = -c_{L-1}^{\dagger}c_0^{\dagger} - c_0c_{L-1} - c_{L-1}^{\dagger}c_0 - c_0^{\dagger}c_{L-1} \tag{20}$$

This expression can be compared with the equation obtained by formally writing down Eq. (6) for the case $i = L - 1$:

$$\sigma_{L-1,j+1}\sigma_{L,j+1} = c_{L-1}^{\dagger}c_L^{\dagger} + c_Lc_{L-1} + c_{L-1}^{\dagger}c_L + c_L^{\dagger}c_{L-1} \tag{21}$$

Thus, we can introduce a second quantized operator for site L provided that we satisfy the condition

$$c_L \equiv -c_0 \tag{22}$$

The sign in Eq. (22) becomes positive for odd states. These, however, are not important in the temperature region $T > T_c$, where T_c is the critical temperature. Therefore, we ignore this alternative possibility in the present work. Our principal interest is the study of the onset of long-range correlation as the critical point is approached from above. Equation (22) provides a kind of periodic boundary condition on the expansion of the second quantized operators in terms of their Fourier components. Introducing the sum over the wave number q and normalizing to the total width L, we have the second quantized operators at site ν given by the expansions

$$c_\nu = L^{-1/2}\sum_q e^{i\nu q}c_q \tag{23}$$

and

$$c_\nu^{\dagger} = L^{-1/2}\sum_q e^{-i\nu q}c_q^{\dagger} \tag{24}$$

Equation (22) fixes the discrete values taken on by q as the half-odd integer multiples of the interval

$$\Delta q = 2\pi/L \tag{25}$$

(In the case of the odd states, the quantization of q is specified as integer multiples of Δq.)

It is useful to study separately the exchange terms in Eq. (18). These are diagonal in the excitation number operator n. Summing over all sites, we find

$$\begin{aligned}\sum_{\nu} c_{\nu}^{\dagger} c_{\nu+1} &= L^{-1} \sum_{qq'} c_q^{\dagger} c_{q'} \sum_{\nu} \exp[i\nu(q' - q) + iq'] \\ &= \sum c_q^{\dagger} c_{q'} (\exp iq')\, \delta_{q,q'} \\ &= \sum_{q} n_q \exp iq \end{aligned} \tag{26}$$

where we write the number operator for wave number mode q as $n_q \equiv c_q^{\dagger} c_q$. Note that $n = \sum_q n_q$. Similarly, we obtain

$$\sum_{\nu} c_{\nu+1}^{\dagger} c_{\nu} = \sum_{q} n_q e^{-iq} \tag{27}$$

Ignoring for the moment the perturbing effect of the creation and annihilation of pairs [i.e., the other two terms in Eq. (18)], we obtain as a first approximation to the portion of the transfer matrix associated with the horizontal bonds

$$M = \exp\left(2J \sum_{q} n_q \cos q\right) \tag{28}$$

With

$$M' = \exp\left(-2\bar{J}' \sum_{q} n_q\right) \tag{29}$$

we find for the complete transfer matrix in this approximation the product

$$MM' = \prod_{q} \exp[-2n_q(\bar{J}' - J \cos q)] \tag{30}$$

The thermodynamic problem of the evaluation of the partition function is equivalent to finding the largest eigenvalue of the transfer matrix. From Eq. (30), we see that this is given by no excitation (all $n_q = 0$), provided that $\bar{J}' > J$. But as the temperature is lowered, J increases and $\bar{J}'$ decreases, so that eventually an instability will set in at $q = 0$. This establishes the critical point as

$$\bar{J}_c' = J_c \tag{31}$$

Substitution from Eq. (17) yields the Kramers–Wannier equation

$$\sinh 2J_c \cdot \sinh 2J_c' = 1 \tag{32}$$

In the case of an isotropic lattice where $J' = J$, Eq. (32) requires $\sinh 2J_c = 1$, or $J_c = 0.441$.

3. PAIR INTERACTIONS

Now we study the perturbing effect of the pair creation and annihilation processes. Summing over all sites gives

$$\begin{aligned}\sum_\nu c_{\nu+1}c_\nu &= L^{-1}\sum_{qq'} c_q c_{q'} \sum_\nu \exp[i\nu(q+q') + iq] \\ &= \sum_{qq'} c_q c_{q'}(\exp iq)\,\delta_{q,-q'} = \sum_q c_q c_{-q} \exp iq \\ &= \sum_{q>0} [(\exp iq) - (\exp -iq)]\, c_q c_{-q} \\ &= 2i \sum_{q>0} (\sin q)\, c_q c_{-q} \approx 2i \sum_{q>0} q c_q c_{-q}\end{aligned} \tag{33}$$

Here, we have taken note of the fact that the terms for negative q are not independent of those for $q > 0$. We have also introduced the approximation $\sin q \approx q$, valid for long wavelengths ($q \ll 1$). Including now the pair processes, we have to return to Eq. (28) and write the factor associated with the horizontal bonds more completely as

$$M = \exp\left(2J_n + \sum_q \lambda_q \mathcal{O}_q\right) = \exp(2J_n + \mathcal{O}) \tag{34}$$

where

$$\mathcal{O} = \sum_q \lambda_q \mathcal{O}_q \tag{35}$$

$$\mathcal{O}_q = i(c_q c_{-q} - c_{-q}^\dagger c_q^\dagger) \tag{36}$$

and

$$\lambda_q = 2Jq \tag{37}$$

It is convenient to write the combined transfer operator, including the effects of both the horizontal and vertical bonds, in the symmetrized form

$$(M')^{1/2}\, M (M')^{1/2} = [\exp(-\bar{J}'n)]\, M \exp(-\bar{J}'n) \tag{38}$$

In the $q = 0$ limit, where $\mathcal{O}$ can be neglected, M and M' commute and Eq. (38) reduces to Eq. (32). With the approximation $\cos q \approx 1$, which has already been used in Eq. (34), this can be written in the simplified form

$$\exp[-2n(\bar{J}' - J)] \approx 1 - 2n(\bar{J}' - J)$$

Thus, in the vicinity of the critical point, $\bar{J}' - J \approx 0$, and the transfer matrix differs only slightly from unity, though this is not true for the individual factors in Eq. (38). This makes possible a continuum treatment of the problem. We now have to calculate the change in M produced by $\mathcal{O}$ treated as a first-order perturbation. This approach is justified by the fact that the long wavelengths dominate in the approach to the critical point. For these, the "coupling constants" are $\lambda_q \ll 1$.

In calculating M to first order in $\mathcal{O}$, the lack of commutativity between the operators $\mathcal{O}$ and n is an essential complication. This is a familiar problem in quantum field theory and can be handled here by dividing M up into a very large number N of equal factors, each equal to $M^{1/N} \approx 1 + 2Jn/N + \mathcal{O}/N$. We can set $\mathcal{O}$ equal to zero in all of the factors except one, which gives

$$\begin{aligned} M &= \lim_{N\to\infty} (M^{1/N})^N \\ &= \lim_{N'\to\infty} \left(1 + \frac{2Jn}{N}\right)^N + \lim_{N\to\infty} \sum_{N'=1}^{N} \left(1 + \frac{2Jn}{N}\right)^{N-N'} \frac{\mathcal{O}}{N} \left(1 + \frac{2Jn}{N}\right)^{N'-1} \\ &= e^{2Jn} + \int_0^1 dx\, e^{2Jn\{1-x\}} \mathcal{O} e^{2Jnx} \\ &= e^{2Jn} + (1/2J) \int_{-J}^{+J} dy\, e^{Jn} e^{-ny} \mathcal{O} e^{ny} e^{Jn} \\ &= e^{2Jn} + (1/2J)\, e^{Jn} \int_{-J}^{+J} dy\, \mathcal{O}_{\text{int}}(y)\, e^{Jn} \end{aligned} \tag{39}$$

where

$$\mathcal{O}_{\text{int}}(y) \equiv e^{-ny} \mathcal{O} e^{ny} \tag{40}$$

is a kind of "interaction picture" operator. Note that $\mathcal{O}_{\text{int}}(y)$ is not Hermitian, but instead satisfies the adjoint condition $\mathcal{O}_{\text{int}}(y)^\dagger = \mathcal{O}_{\text{int}}(-y)$. When we substitute Eq. (40) into Eq. (38), we can make the approximation $\bar{J}' = J$ in the second term (which is already small), to obtain

$$\begin{aligned} (M')^{1/2} M (M')^{1/2} &= \exp[-2(\bar{J}' - J)\, n] + (1/2J) \int_{-J}^{+J} dy\, \mathcal{O}_{\text{int}}(y) \\ &\approx 1 - 2(\bar{J}' - J)\, n + (1/2J) \int_{-J}^{+J} dy\, \mathcal{O}_{\text{int}}(y) \\ &\approx \exp\left[-2(\bar{J}' - J)\, n + (1/2J) \int_{-J}^{+J} dy\, \mathcal{O}_{\text{int}}(y)\right] \end{aligned} \tag{41}$$

It is easy to see that the annihilation and creation terms of $\mathcal{O}$ acquire factors $e^{\pm 2y}$, respectively, by virtue of the transformation of Eq. (40). I.e.,

$$\mathcal{O}_{q,\mathrm{int}}(y) = ie^{2y}c_q c_{-q} - ie^{-2y}c_{-q}^{\dagger}c_q^{\dagger} \tag{42}$$

so that the integration yields

$$\int_{-J}^{+J} \mathcal{O}_{q,\mathrm{int}}(y)\, dy = (\sinh 2J)\, \mathcal{O}_q \tag{43}$$

Substitution into Eq. (41) gives

$$\frac{1}{2J}\int_{-J}^{+J} dy\, \mathcal{O}_{\mathrm{int}}(y) = \frac{\sinh 2J}{2J}\sum_{q>0}\lambda_q \mathcal{O}_q = (\sinh 2J)\sum_{q>0} q\mathcal{O}_q\,. \tag{44}$$

Equation (44), when substituted into Eq. (41), solves the problem of expressing the transfer matrix in terms of the small operators $2(\bar{J}' - J)\,n$ and $q\mathcal{O}_q$. A minor defect of this result is that it depends upon the exchange constant J through the factor $\sinh 2J$. A more satisfactory J-independent universal continuum formulation is obtained by changing the vertical scale. We introduce a continuum coordinate t in place of the discrete variable j. Measured in this new coordinate, the spacing of the horizontal rows is taken to be

$$\Delta t = \sinh 2J \tag{45}$$

which permits us to write the transfer matrix in the universal form

$$(M')^{1/2}\, M (M')^{1/2} = e^{-(\Delta t)H} \tag{46}$$

with the "Hamiltonian operator"

$$H = \kappa n - \sum_{q>0} q\mathcal{O}_q \tag{47}$$

and the reciprocal correlation length

$$\kappa = 2(\bar{J}' - J)/\sinh 2J \tag{48}$$

4. DIRAC EQUATION

Just as we have replaced the discrete variable j by the continuous variable t, we can replace the other lattice coordinate i by the continuous variable x. Integrating over x, we find the identity

$$\int_0^L dx\, \phi(x)\, \nabla\phi(x) = -2i\sum_{q>0} q c_q c_{-q} \tag{49}$$

so that we can equally well express the "Hamiltonian" by the integral

$$H = \int dx\, H(x) + \tfrac{1}{2}\kappa L \tag{50}$$

where the "Hamiltonian density" is

$$H(x) = \tfrac{1}{2}(\phi\, \nabla\phi - \phi^\dagger\, \nabla\phi^\dagger) + \kappa(\phi^\dagger\phi - \tfrac{1}{2}) \tag{51}$$

We are now writing the continuous fermion field operator as $\phi(x)$ in place of the site operator c_ν of Eq. (23). The gradient and nongradient terms of Eq. (51) are reminiscent of the Dirac equation[15] written in one space dimension with a Hamiltonian density of the form

$$H_D(x) = -i\psi^\dagger\alpha\, \nabla\psi + \kappa\psi^\dagger\beta\psi \tag{52}$$

The particle mass is κ and we chose the units so that the velocity of light is unity. The Dirac matrices satisfy the conditions

$$\alpha\beta + \alpha\beta = 0 \tag{53a}$$

$$\alpha^2 = \beta^2 = 1 \tag{53b}$$

These equations are fulfilled by

$$\alpha = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{54a}$$

$$\beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{54b}$$

where σ_i are the Pauli spin matrices. Thus, $\psi(x)$, the second quantized Dirac field in one dimension, has only two components. These are required for describing the positive- and negative-energy states. The further spin degeneracy which appears in three dimensions is not required in one dimension.

Because of the two-component nature of $\psi(x)$, it is not possible to transform Eq. (51) directly into Eq. (52). It is first necessary to double the number of degrees of freedom in the Ising model. This we accomplish by simply imagining that there is a second Ising lattice superposed on the first but not interacting with it. We then obtain a two-component Ising field $\phi_{1,2}$ which can be identified with the Dirac field ψ. The desired connection is given by[3]

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_1^\dagger + i\phi_2^\dagger \end{pmatrix} \tag{55}$$

[3]This form for ψ is somewhat related to Nambu's[16] two-component formulation of superconductivity theory.

The different components of the Ising field are assumed to anticommute with one another and to satisfy the canonical anticommutation relations

$$\{\phi_i(x), \phi_j^\dagger(x')\} = \delta_{ij}\,\delta(x - x') \tag{56a}$$

$$\{\phi_i(x), \phi_j(x')\} = 0 \tag{56b}$$

We verify that the components of ψ also satisfy the canonical anticommutation relations, by virtue of Eqs. (55) and (56a, b). We now confirm that the gradient terms in Eqs. (52) and (53) match exactly:

$$\begin{aligned} -i\psi^\dagger\alpha\,\nabla\psi &= \psi^\dagger \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla\psi \\ &= \tfrac{1}{2}(\phi_1\nabla\phi_1 - \phi_1^\dagger\nabla\phi_1^\dagger) + \tfrac{1}{2}(\phi_2\nabla\phi_2 - \phi_2^\dagger\nabla\phi_2^\dagger) \\ &\quad + \tfrac{1}{2}i(\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1 + \phi_2^\dagger\nabla\phi_1^\dagger - \phi_1^\dagger\nabla\phi_2^\dagger) \\ &= \tfrac{1}{2}(\phi_1\nabla\phi_1 - \phi_1^\dagger\nabla\phi_1^\dagger) + \tfrac{1}{2}(\phi_2\nabla\phi_2 - \phi_2^\dagger\nabla\phi_2^\dagger) \\ &\quad + \tfrac{1}{2}i\,\nabla(\phi_1\phi_2 + \phi_2^\dagger\phi_1^\dagger) \end{aligned} \tag{57}$$

The last term in Eq. (57) is a perfect derivative and consequently disappears in the space integration. It does not contribute to the total energy. The remaining terms are precisely the desired gradient terms of Eq. (51). The mass term in Eq. (52) is

$$\begin{aligned} \psi^\dagger\beta\psi &= \psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi \\ &= \tfrac{1}{2}[\phi_1^\dagger, \phi_1] + \tfrac{1}{2}[\phi_2^\dagger, \phi_2] + \tfrac{1}{2}i\{\phi_1^\dagger, \phi_2\} - \tfrac{1}{2}i\{\phi_2^\dagger, \phi_1\} \end{aligned} \tag{58}$$

The anticommutators vanish by Eq. (56a), while application of Eq. (56a) to the commutators gives

$$\psi^\dagger\beta\psi = \phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 - \delta(0) \tag{59}$$

The delta function is smeared out over a lattice spacing because of the cutoff of q at the maximum wave number π [see Eq. (25)]. Therefore $\delta(0)$ is equal to unity rather than to infinity and the correspondence to Eq. (51) is complete [i.e., $\int dx(H\{\phi_1\} + H\{\phi_2\}) = \int dx\, H_D(x)$].

This completes our identification of the two-dimensional Ising model with the Dirac equation for spinless fermions moving freely without interaction in one-dimensional space.[4] From Eq. (46), we see that the largest eigenvalue of the transfer matrix will be determined by the ground-state eigenvalue E_G of the Dirac Hamiltonian

$$H_D \equiv \int dx\, H_D(x) \tag{60}$$

[4] For an alternative equivalence, see Suzuki.[17]

This is obtained by filling up all of the negative-energy single-particle states,

$$E_G = -\sum_q (\kappa^2 + q^2)^{1/2} = -\frac{L}{\pi}\int_0^{q_c} dq\,(\kappa^2 + q^2)^{1/2} \tag{61}$$

Here, we have replaced the upper limit π by an effective cutoff q_c, since the long wavelengths which underlie the continuum approximation are not accurate for q values of the order of magnitude of unity.

5. CRITICAL SPECIFIC HEAT

The exchange constant and the temperature enter the continuum model only through the correlation length by virtue of Eq. (48). Thus, changes in J_c can be absorbed by corresponding changes in the temperature scale. The normalized derivative of κ evaluated at the transition temperature is

$$\begin{aligned} T\kappa' \equiv T\frac{d\kappa}{dT}\bigg|_{T_c} &= -J\frac{d\kappa}{dJ}\bigg|_{J_c} \\ &= \frac{2}{\sinh 2J_c}\left(J - J\frac{d\bar{J}'}{dJ}\right)_c = \frac{2}{\sinh 2J_c}\left(J - J'\frac{d\bar{J}'}{dJ'}\right)_c \end{aligned} \tag{62}$$

With

$$\bar{J}' = -\tfrac{1}{2}\ln\tanh J' \tag{63}$$

from Eq. (16), we find

$$d\bar{J}'/dJ' = -1/\sinh 2J' \tag{64}$$

Since two temperature differentiations are required to find the specific heat, we are interested in the square of $T\kappa'$. Including the scale factor of Eq. (45) and substituting from Eq. (64) gives the characteristic constant

$$\begin{aligned} C_0 &= (\sinh 2J_c)(T\kappa')^2 \\ &= 4\left(J_c + \frac{J_c'}{\sinh 2J_c'}\right)\left(J_c' + \frac{J_c}{\sinh 2J_c}\right) \end{aligned} \tag{65}$$

symmetric under the exchange $J_c \leftrightarrow J_c'$. For an isotropic lattice, $\sinh 2J_c = 1$,

$$J_c = 0.441 \qquad \text{and} \qquad C_0 = 3.11$$

To get the free energy per site, we have to divide the total ground-state energy (61) by the total number of vertical lines $2L$ (for the two noninteracting lattices). Therefore, the entropy is proportional to the derivative

$$-\frac{dE_G/2L}{d\kappa} = \frac{\kappa}{2\pi}\int_0^{q_c}\frac{dq}{(\kappa^2 + q^2)^{1/2}} \tag{66}$$

The constant term $-\frac{1}{2}$ from (50) has been neglected. The singular part of the second derivative is

$$-\frac{d^2(E_G/2L)}{d\kappa^2} = \frac{1}{2\pi}\int_0^{q_c}\frac{dq}{(\kappa^2+q^2)^{1/2}} = \frac{1}{2\pi}\ln\frac{2q_c}{\kappa} \tag{67}$$

Consequently, the singular specific heat is

$$C = -C_0\frac{d^2(E_G/2L)}{d\kappa^2} = \frac{C_0}{2\pi}\ln\frac{2q_c}{\kappa} = \frac{C_0}{2\pi}\ln\frac{2q_c}{T_c\kappa'\cdot(T-T)/T_c} \tag{68}$$

exhibiting the Onsager logarithmic singularity, corresponding to the critical index $\alpha = 0$. The additive constant is undetermined in the present continuum solution. The multiplicative constant for an isotropic lattice equals $C_0/2\pi = 0.495$.

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